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RECAPTURING H^2-FUNCTIONS ON A POLYDISC

BY

D. J. PATIL

ABSTRACT. Let U^2 be the unit polydisc and T^2 its distinguished boundary. If $E \subset T^2$ is a set of positive measure and the restriction to E of a function f in $H^2(U^2)$ is given then an algorithm to recapture f is developed.

Introduction. Let U be the open unit disc in the complex plane and T its boundary. Let f be holomorphic in the unit polydisc $U^2 = U \times U$; then $f(z) = \sum c(n)z^n$, $z = (z_1, z_2) \in U^2$, $n = (n_1, n_2) \in Z^2_+$. The function f is in H^2 if and only if $\sum |c(n)|^2 < \infty$ [2, p. 50]. The functions f in H^2 can be identified with the boundary value functions on the distinguished boundary T^2 $(= T \times T)$ of U^2 and these boundary value functions are precisely those $f \in L^2(T^2)$ whose Fourier coefficients $\hat{f}(n_1, n_2)$ are zero if either $n_1 < 0$ or $n_2 < 0$. It is known that if a nonzero f is in H^2 then $\log|f|$ is in $L^1(T^2)$ and hence if $f = 0$ on a subset E of T^2 of positive measure then f is the zero function. It follows that the restriction to such a set E of a function f in H^2 determines f uniquely. Following the methods in [1], we give a constructive algorithm to recapture the function f from its values on E. The construction is in two steps. From the knowledge of f on E, the first step obtains f on $F \times T$ where F is some subset of T of positive measure. The second step is the 'conjugate' of the first and starting with f on $F \times T$ we recover f on the whole of $T \times T$.

The arrangement of the paper is as follows. After defining the notations, we prove some lemmas leading to Theorem 1 which gives the first step in recapturing f. We then discuss how the second step is a corollary of the first. This is followed by Theorem 2 which makes the algorithm more explicit.

Notations. In the following H^2 will stand for $H^2(T^2)$. By L^2 and L^∞ will be meant $L^2(T^2)$ and $L^\infty(T^2)$ respectively. The subspace L^2_+ (L^∞_+) will consist of those f in L^2 (L^∞) whose Fourier coefficients vanish in the lower half plane, thus, $L^2_+ = \{f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for all } n_2 < 0\}$. The orthogonal projection of L^2 onto L^2_+ will be denoted by P. The Toeplitz operator T_ϕ corresponding to $\phi \in L^\infty$ is defined by $T_\phi f = P(\phi f)$, $f \in L^2_+$. If $f \in L^2_+$, we may consider f as an H^2-function on T with values in $L^2(T)$: $f(\theta_1, \theta_2) = \sum_{n \geq 0} f_n(\theta_1)e^{in\theta_2}$. The natural

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extension of this f to $\overline{U}$ will also be denoted by f; then for $z_2 \in U$, $f(z_2) = \sum_{n \geq 0} f_n(\theta_1) z_2^n$. Such functions f are holomorphic $L^2(T)$-valued functions on U and the relation $f(r_2 e^{i\theta_2}) \to f(\theta_1, \theta_2)$ as $r_2 \to 1$ holds for a.e. pointwise convergence and also for mean convergence [3, p. 186]. Thus the functions $f(\theta_1, \theta_2)$ and $f(z_2)$ above determine each other. The (normalized) Haar measures on T and T^2 will be denoted by m_1 and m_2 respectively, and when there is no risk of confusion $dm_1(\theta_1)$ will be shortened to $d\theta_1$ etc. For $z \in U$, $\theta \in T$, let $K(z, \theta) = (e^{i\theta} + z)/(e^{i\theta} - z)$ and $P = \operatorname{Re} K$.

Lemma 1. *Let $E \subset T^2$, $m_2(E) > 0$. For $\lambda > 0$ and $(z_1, z_2) \in U^2$ define*

$$t_\lambda(z_1, z_2) = \exp\left\{\tfrac{1}{2}\log(1+\lambda)\iint_E P(z_1, \theta_1) K(z_2, \theta_2)\, d\theta_1\, d\theta_2\right\}.$$

Then we have the following:

(a) *For all $(z_1, z_2) \in U^2$, $1 \leq |t_\lambda(z_1, z_2)| \leq \sqrt{(1+\lambda)}$.*

(b) *There exists a set $A \subset T$ with $m_1(A) = 1$ such that for every $\theta_1 \in A$ and for every $z_2 \in U$, the limit*

$$t_\lambda(\theta_1, z_2) = \lim_{r_1 \to 1} t_\lambda(r_1 e^{i\theta_1}, z_2)$$

exists and for each $\theta_1 \in A$, the function $t_\lambda(\theta_1, \cdot)$ is holomorphic in U.

(c) *For each $\theta_1 \in A$, the limit*

$$t_\lambda(\theta_1, \theta_2) = \lim_{r_2 \to 1} t_\lambda(\theta_1, r_2 e^{i\theta_2})$$

exists for almost all θ_2 in T, and hence $t_\lambda(\theta_1, \theta_2)$ is defined almost everywhere in T^2 and is in L_+^∞.

(d) *For almost all (θ_1, θ_2) in T^2,*

$$|t_\lambda(\theta_1, \theta_2)|^2 = 1 + \lambda \chi_E(\theta_1, \theta_2).$$

Proof. (a) holds since $|t_\lambda(z_1, z_2)|$ is the exponential of the Poisson integral of $\frac{1}{2}\log(1+\lambda)\chi_E$.

(b) For $z_2 \in U$ and $\theta_1 \in T$, letting

$$\mathcal{B}(\theta_1, z_2) = \int_T \chi_E(\theta_1, \theta_2) K(z_2, \theta_2)\, d\theta_2,$$

we see that $|\mathcal{B}(\theta_1, z_2)| \leq 2/(1 - |z_2|)$, for all $\theta_1 \in T$ and hence by Fatou's theorem the integral

$$\iint_E P(r_1 e^{i\alpha_1}, \theta_1) K(z_2, \theta_2)\, d\theta_1\, d\theta_2 = \int_T P(r_1 e^{i\alpha_1}, \theta_1) \mathcal{B}(\theta_1 z_2)\, d\theta_1$$

converges to $\mathcal{B}(\alpha_1, z_2)$ a.e. (α_1) as $r_1 \to 1$. Therefore

(1) $$t_\lambda(r_1e^{i\theta_1}, z_2) \to \exp\{\tfrac{1}{2}\log(1+\lambda)\mathcal{B}(\theta_1, z_2)\}$$

a.e. (θ_1). The set of measure 1 where this convergence takes place depends on z_2. To see that there is a set A of measure 1 such that for all $\theta_1 \in A$ and for all $z_2 \in U$, (1) holds it suffices to observe that for a fixed α_1, the family $\{F_{r_1} : 0 < r_1 < 1\}$ where

$$F_{r_1}(z_2) = \int_T P(r_1e^{i\alpha_1}, \theta_1)\mathcal{B}(\theta_1, z_2)\,d\theta_1,$$

is equicontinuous on compact subsets of U. A standard argument such as the one in the proof of (5.16) in [5, p. 327] now proves the existence of the set A.

(c) For each $\theta_1 \in A$, $t_\lambda(\theta_1, \cdot)$ is a bounded holomorphic function on U and hence by Fatou's theorem $t_\lambda(\theta_1, r_2e^{i\theta_2})$ has a limit—say $t_\lambda(\theta_1, \theta_2)$—as $r_2 \to 1$, a.e. (θ_2). Since $t_\lambda(\theta_1, \theta_2)$ is a repeated limit of continuous functions, its domain Δ is a measurable set in T^2 and since almost every θ_1-section of Δ has measure 1, we must have that $m_2(\Delta) = 1$.

(d) We have for almost every $(\theta_1, \theta_2) \in T^2$,

$$|t_\lambda(\theta_1, \theta_2)|^2 = \lim_{r_2 \to 1} \exp\left\{\log(1+\lambda)\int_T \chi_E(\theta_1, \alpha_2)P(r_2e^{i\theta_2}, \alpha_2)\,d\alpha_2\right\}$$

$$= \exp\{\log(1+\lambda)\chi_E(\theta_1, \theta_2)\} = 1 + \lambda\chi_E(\theta_1, \theta_2).$$

Lemma 2. *Let $E \subset T^2$, $m_2(E) > 0$. Let S be the Toeplitz operator on L^2_+ corresponding to the characteristic function χ_E of the set E. If t_λ is as in Lemma 1 and $s_\lambda = 1/t_\lambda$, then $(I + \lambda S) = T_{\bar{t}_\lambda}T_{t_\lambda}$, $(I+\lambda S)^{-1} = T_{s_\lambda}T_{\bar{s}_\lambda}$.*

Proof. We observe that $s_\lambda \in L^\infty_+$. The rest of the proof is similar to those of Lemmas 1, 2 in [1].

Lemma 3. *Let $s(\theta_1, \theta_2) \sim \sum_{n\ge 0} s_n(\theta_1)e^{in\theta_2}$ be in L^∞_+ and let $s(\theta_1, z_2) = \sum_{n\ge 0} s_n(\theta_1)z_2^n$, $\theta_1 \in T$, $z_2 \in U$. Define for $(\theta_1, \theta_2) \in T^2$ and $z \in U$,*

$$e_z^{(j)}(\theta_1, \theta_2) = (1 - \bar{z}e^{i\theta_j})^{-1}, \quad j = 1, 2.$$

Then for $z_1, z_2 \in U$, $e_{z_1}^{(1)}e_{z_2}^{(2)}$ is in H^2 (and so in L^2_+) and

$$T_{\bar{s}}(e_{z_1}^{(1)}e_{z_2}^{(2)})(\theta_1, \theta_2) = \bar{s}(\theta_1, z_2)(e_{z_1}^{(1)}e_{z_2}^{(2)})(\theta_1, \theta_2).$$

Proof. The equality follows by checking that the inner products of both sides with $\exp(im_1\theta_1 + im_2\theta_2)$, are the same for every integer m_1 and every nonnegative integer m_2.

Lemma 4. *Let $E \subset T^2$ with $m_2(E) > 0$. Let for each $\theta_1 \in T$, $E(\theta_1) = \{\theta_2 \in T: (\theta_1, \theta_2) \in E\}$ and let, for each $\delta > 0$, $F(\delta) = \{\theta_1 \in T: m_1(E(\theta_1)) > \delta\}$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that $m_1(F(\delta_2)) > \delta_1$.*

Proof. Observe that as $n \to \infty$,

$$\int_{F(1/n)} m_1(E(\theta_1))\, d\theta_1 \uparrow \int_T m_1(E(\theta_1))\, d\theta_1 = m_2(E),$$

and hence, if $0 < \delta_1 < m_2(E)$, there exists N such that

$$\delta_1 < \int_{F(1/N)} m_1(E(\theta_1))\, d\theta_1 \le m_1(F(1/N)).$$

Now take $\delta_2 = 1/N$.

Theorem 1. *Let $E \subset T^2$, $m_2(E) > 0$. Choose δ_1, δ_2 as in Lemma 4 and denote $F(\delta_2)$ by F. Let M be the operator defined by*

$$(Mf)(\theta_1, \theta_2) = \chi_F(\theta_1) f(\theta_1, \theta_2), \qquad f \in L_+^2,\ (\theta_1, \theta_2) \in T^2.$$

Suppose that S is as in Lemma 2. Then for every $f \in H^2$,

$$\lim_{\lambda \to \infty} \lambda M(I + \lambda S)^{-1} S f = Mf.$$

Proof. Since $M(I - \lambda(I + \lambda S)^{-1} S) = M(I + \lambda S)^{-1}$, and the set $\{e_{z_1}^{(1)} e_{z_2}^{(2)}: z_1, z_2 \in U\}$ is fundamental in H^2, the theorem will be proved if the following are verified: (i) $\sup_\lambda \|M(I + \lambda S)^{-1}\| < \infty$, and (ii) for all $z_1, z_2 \in U$, $M(I + \lambda S)^{-1} e_{z_1}^{(1)} e_{z_2}^{(2)} \to 0$, as $\lambda \to \infty$.

By Lemma 1(a), $|s_\lambda(\theta_1, \theta_2)| \le 1$, a.e. and therefore $\|T_{\bar{s}_\lambda}\| = \|T_{s_\lambda}\| \le \|s_\lambda\|_\infty \le 1$. Also trivially, $\|M\| \le 1$ and hence in view of Lemma 2, $\|M(I + \lambda S)^{-1}\| = \|M T_{s_\lambda} T_{\bar{s}_\lambda}\| \le 1$. This proves (i).

To check (ii), note that for $\theta_1 \in F$, using the notation in the proof of Lemma 1(b),

$$\operatorname{Re} \mathcal{B}(\theta_1, z_2) = \int_{E(\theta_1)} P(z_2, \theta_2)\, d\theta_2 \ge \frac{1 - |z_2|}{1 + |z_2|} \delta_2,$$

and hence for almost all $\theta_1 \in F$,

$$|s_\lambda(\theta_1, z_2)| \le (1 + \lambda)^{-p}, \qquad z_2 \in U$$

where $p = \delta_2(1 - |z_2|)/2(1 + |z_2|) > 0$. Now using Lemma 3, we have

$$M(I + \lambda S)^{-1}(e_{z_1}^{(1)} e_{z_2}^{(2)})(\theta_1, \theta_2) = \chi_F(\theta_1) s_\lambda(\theta_1, \theta_2) \bar{s}_\lambda(\theta_1, z_2)(e_{z_1}^{(1)} e_{z_2}^{(2)})(\theta_1, \theta_2),$$

and hence

$$\|M(I+\lambda S)^{-1}(e_{z_1}^{(1)}e_{z_2}^{(2)})\|_2 \le (1+\lambda)^{-p}\|e_{z_1}^{(1)}e_{z_2}^{(2)}\|_2.$$

This last expression tends to zero as $\lambda \to \infty$ and (ii) is verified.

Discussion. The limit relation $\lambda M(I+\lambda S)^{-1}Sf \to Mf$ of Theorem 1 provides the first step in recapturing f from its values on E. The knowledge of f on E yields Sf and $(I+\lambda S)^{-1}$ is obtained in terms of s_λ which depends only on the set E. Thus $\lambda M(I+\lambda S)^{-1}Sf$ can be computed and the limit as $\lambda \to \infty$ gives Mf, i.e. values of f on $F \times T$. The second step of going from $F \times T$ to $T \times T$ now follows easily. We proceed basically as in the first step, but we interchange the roles of the θ_1- and θ_2-coordinates and employ the appropriate substitute for L_+^2. The original set E is now replaced by $E' = F \times T$. Recalling the relationship that F bears to E (Lemma 4), we see that the set F' which corresponds in a similar way (but with θ_1, θ_2 interchanged) to E' can be chosen to be T itself! Thus a theorem such as Theorem 1 with suitable changes will lead us to f on $T \times F' = T \times T$. This is the sought-for second step, and the algorithm for recapturing f is complete.

The above algorithm recaptures the boundary values of f from its values on E. It is sometimes convenient to have a formula which gives the values of f inside U^2 directly. In the following theorem such a formula is obtained.

Theorem 2. *Let the hypotheses and the notations be as in Theorem* 1.

(a) *If for $\lambda > 0$ and $z_1, z_2 \in U$,*

$$f_\lambda(z_1, z_2) = \frac{\lambda}{(2\pi i)^2}\iint_E \frac{f(w_1, w_2)\bar{s}_\lambda(w_1, w_2)s_\lambda(w_1, z_2)\chi_F(w_1)}{(w_1 - z_1)(w_2 - z_2)}\, dw_1\, dw_2,$$

then as $\lambda \to \infty$, f_λ converges in H^2 to some ϕ and a fortiori uniformly on compact subsets of U^2.

(b) *If for $\lambda > 0$ and $z_1 \in U$,*

$$h_\lambda(z_1) = \exp\left\{-\tfrac{1}{2}\log(1+\lambda)\int_F K(z_1, \theta_1)\, d\theta_1\right\},$$

and ϕ is as in (a) *above then for each $(z_1, z_2) \in U^2$,*

$$f(z_1, z_2) = \lim_{\lambda\to\infty}\lim_{r\to 1}\frac{\lambda}{2\pi i}h_\lambda(z_1)\int_{c_r}\frac{\phi(w_1, z_2)\bar{h}_\lambda(w_1)\, dw_1}{w_1 - z_1},$$

where c_r is the circle $|w_1| = r$ and $|z_1| < r < 1$.

Proof. (a) By Theorem 1, we have that as $\lambda \to \infty$, $\lambda M(I+\lambda S)^{-1}Sf \to Mf$. Taking the inner product with $e_{z_1}^{(1)}e_{z_2}^{(2)}$, $z_1, z_2 \in U$, we get

(2) $$\lambda(M(I+\lambda S)^{-1}Sf,\ e^{(1)}_{z_1}e^{(2)}_{z_2}) \to (Mf,\ e^{(1)}_{z_1}e^{(2)}_{z_2}).$$

We will prove that the first member of (2) equals $f_\lambda(z_1, z_2)$ and hence if the second member of (2) is denoted by $\phi(z_1, z_2)$ the proof of (a) will be complete.

Since M is the multiplication by $\chi_F(\theta_1)$, a function of θ_1 only, M commutes with S and therefore with $(I+\lambda S)^{-1}$. Thus

$$(M(I+\lambda S)^{-1}Sf,\ e^{(1)}_{z_1}e^{(2)}_{z_2}) = (SMf,\ (I+\lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2}) = (Mf,\ (I+\lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2})_E.$$

Using Lemmas 2 and 3 to write the expressions for $(I+\lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2}$, we see that the left member of (2) is in fact $f_\lambda(z_1, z_2)$.

(b) From the proof in (a), we see that $\phi(z_1, z_2) = (Mf,\ e^{(1)}_{z_1}e^{(2)}_{z_2})$ and hence

$$\phi(z_1, z_2) = \frac{1}{(2\pi i)^2}\iint_{F\times T}\frac{f(w_1, w_2)}{(w_1 - z_1)(w_2 - z_2)}\,dw_1\,dw_2.$$

Let us now define L^2_- to be the subspace $\{f \in L^2 : \hat{f}(m, n) = 0 \text{ for all } m < 0\}$, $\tilde{P}$ the orthogonal projection of L^2 onto L^2_-, $\tilde{S}$ the Toeplitz operator on L^2_- corresponding to the function $\chi_{E'}$, where $E' = F\times T$, i.e. $\tilde{S}(f) = \tilde{P}(\chi_{E'}f)$, $f \in L^2_-$, and σ_λ, b_λ to be

$$\sigma_\lambda(z_1, z_2) = \exp\left\{-\tfrac{1}{2}\log(1+\lambda)\iint_{E'}K(\theta_1, z_1)P(\theta_2, z_2)\,d\theta_1\,d\theta_2\right\},$$

$$b_\lambda(z_1) = \exp\left\{-\tfrac{1}{2}\log(1+\lambda)\int_F K(\theta_1, z_1)\,d\theta_1\right\},$$

where $\lambda > 0$ and $z_1, z_2 \in U$. Then for all $z_1, z_2 \in U$, $\sigma_\lambda(z_1, z_2) = b_\lambda(z_1)$ and as in Theorem 1 we will get, for every $f \in H^2$, $\lambda\tilde{M}(I+\lambda\tilde{S})^{-1}\tilde{S}f \to \tilde{M}f$ $(\lambda \to \infty)$, where $\tilde{M}$ is the multiplication by $\chi_{F'}(\theta_2)$, F' corresponding to E' according to Lemma 4 but with roles of θ_1, θ_2 reversed. However, since $E' = F\times T$, we can take $F' = T$ and so for $f \in H^2$, $\lambda(I+\lambda\tilde{S})^{-1}\tilde{S}f \to f$. If, in this last relation, we take the inner product with $e^{(1)}_{z_1}e^{(2)}_{z_2}$, then as $\lambda \to \infty$,

(3) $$(\lambda(I+\lambda\tilde{S})^{-1}\tilde{S}f,\ e^{(1)}_{z_1}e^{(2)}_{z_2}) \to (f,\ e^{(1)}_{z_1}e^{(2)}_{z_2}) = f(z_1, z_2).$$

Now noting that $\tilde{P}(\chi_{E'}f) = \phi$, we see that

$$((I+\lambda\tilde{S})^{-1}\tilde{S}f,\ e^{(1)}_{z_1}e^{(2)}_{z_2}) = (\tilde{S}f,\ (I+\lambda\tilde{S})^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2})$$

$$= (\phi, b_\lambda e^{(1)}_{z_1}e^{(2)}_{z_2})b_\lambda(z_1).$$

In the last step we used results similar to Lemmas 2 and 3 for $(I+\lambda\tilde{S})^{-1}$. From (3) it now follows that as $\lambda \to \infty$,

$$\lambda(\phi, b_\lambda e^{(1)}_{z_1} e^{(2)}_{z_2}) b_\lambda(z_1) \to f(z_1, z_2).$$

It is easy to see that $(\phi, b_\lambda e^{(1)}_{z_1} e^{(2)}_{z_2})$ equals the inner product in $H^2(U^1)$ of $\phi(\cdot, z_2)$ with $b_\lambda e^{(1)}_{z_1}$. The proof is finished by observing that if $u, v \in H^2(U^1)$ and $u_r(e^{i\theta}) = u(re^{i\theta})$ and v_r is similarly defined then the product $(u, v\, e^{(1)}_z)$ is the limit as $r \to 1$ of

$$\frac{1}{2\pi i}\int_{C_r} \frac{u(w)\overline{v}(w)}{w-z}\,dw.$$

Remarks. (1) An alternative proof of a part of the one-variable theorem (Theorem I of [1] as regards the convergence on compact sets) has been suggested by Wainger (see Appendix B of [4]). This proof depends on the following statement which is true if $n = 1$: To each nonnegative function $\psi \in L^\infty(T^n)$ there is $f \in H^\infty(U^n)$ such that $|f| = \psi$ a.e. The statement is false for $n > 1$ [2, p. 54ff]. Moreover for $n > 1$ even when such a function f exists an explicit formula for f does not seem to be known. It would thus appear that a proof on the lines suggested by Wainger is not possible for $n > 1$ and that recourse to the techniques such as the ones used in the present work is necessary.

(2) The Theorem 1 above can easily be generalized to functions in $H^2(T^n)$ with $n > 2$. The algorithm to recapture the function takes n steps and the generalization does not need any new ideas.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MILWAUKEE, MILWAUKEE, WISCONSIN 53201